312. Proposed by J. A. CAPARO, C. E., Notre Dame University, Notre Dame, Ind.

Two roots of the cubic $x^3-px^2+qx-c=0$ are equal. Find their value in terms of p, q, and c.

I. Solution by GEORGE W. HARTWELL, University of Kansas, Lawrence, Kas.; V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.; and G. I. HOPKINS, Instructor in Mathematics and Astronomy, Manchester High School, N. H.

Let m, m, n be the three roots. Then

$$2m+n=p\dots(1);\quad 2mn+m^2=q\dots(2);\quad m^2n=c\dots(3).$$

Solving (1) and (2) for m and n we have,

$$m=\frac{p\pm\sqrt{(p^2-3q)}}{3},\qquad n=\frac{p\mp2\sqrt{(p^2-3q)}}{3}.$$

Substituting these values in (3),

$$(2p^2-6q)\left(\frac{p\pm\sqrt{(p^2-3q)}}{3}\right)=pq-9c\dots(4).\ \text{But}\ \frac{p\pm\sqrt{(p^2-3q)}}{3}=m.$$

Therefore, from (4), $m=\dfrac{pq-9c}{2p^2-6q}$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.; J. SCHEFFER, A. M., Hagerstown, Md.; S. LEFSCHETZ, Wilkinsburg, Pa.; and the PROPOSER.

Let $f(x)=x^3-px^2+qx-c=0$, $f'(x)=3x^2-2px+q=0$.

If $f(x)$ has two equal roots, $f'(x)$ contains one, and hence the greatest common divisor of $f(x)$ and $f'(x)$ gives one of the equal roots. Now if

$$(pq-9c)(27c+4p^3-15pq)=4q(3q-p^2)^2,\ \text{or}$$
$$18cpq+p^2q^2-4cp^3-4q^3-27c^2=0,$$

then $2(3q-p^2)x+pq-9c$ is the greatest common divisor of $f(x)$ and $f'(x)$.

Therefore, $x=(9c-pq)/[2(3q-p^2)]$ is one of the equal roots.

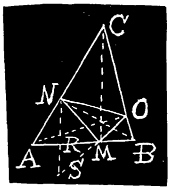
GEOMETRY.

339. Proposed by G. E. BROCKWAY, Boston, Mass.

Of all triangles that can be inscribed in a given triangle, that formed by joining the feet of the altitudes has the minimum perimeter. Prove by means of the straight line and circle.

I. **Solution by J. SCHEFFER, A. M., Hagerstown, Md.**

Assuming the points N and O in the sides AC and BC, $NM+MO$ is a minimum, if $\angle NMA=\angle OMB$; for letting fall the perpendicular NR and extending it to S by its own length, OMS becomes a straight line. It follows from this that in the case of MNO being a triangle of minimum perimeter, $\angle NMA=\angle BMO=\alpha$, $\angle ANM=\angle CNO=\beta$, $\angle MOB=\angle NOC=\gamma$.



Now, $\alpha+\beta=180^\circ-A$, $\alpha+\gamma=180^\circ-B$, $\beta+\gamma=180^\circ-C$.

$\therefore \alpha+\beta+\gamma=\frac{1}{2}(540^\circ-180^\circ)=180^\circ$. $\therefore \alpha=C$, $\beta=B$, $\gamma=A$.

Consequently, the triangle MNO is the pedal triangle.

II. **Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.**

When we wish to find a point P on a given line so that the sum of the distances $PR+PS$ to two given points, is a minimum, it is easy to show that PS and PR must make equal angles with the given line. Let M, L, K be the feet of the altitudes; M on AB, L on AC, K on BC. Then if M, L are fixed, K is the point on BC such that $KL+KM$ is a minimum, since KL, KM make equal angles with BC. Similarly, for M, K and L, K fixed in turn, respectively. $LM+LK$ is a minimum and $ML+MK$ is a minimum for the reason cited above.

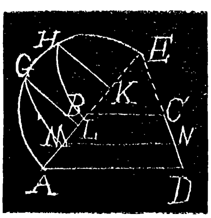
This seems the easiest and simplest proof.

340. **Proposed by J. H. MEYERS, S. J., Sacred Heart College, Augusta, Ga.**

Given trapezoid $ABCD$. Prolong AB and CD, the non-parallel sides, to meet in E. On AE as diameter construct semi-circle $AHGE$. With BE as radius construct arc BG. Draw GK perpendicular to AE. Bisect AH at L. Erect KH perpendicular to AE. Construct arc HM with HE as radius. Draw MN perpendicular to DC. Prove that MN bisects the trapezoid $ABCD$, angles ADC and BCD being right angles.*

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

We may generalize this theorem by drawing the line MN parallel to AD, instead of perpendicular to DC.



$\triangle ADE : \triangle EMN=AE^2 : EM^2=AE^2 : EH^2=AE^2 : AE\times EL=AE : EL$.

$\therefore \triangle ADE-\triangle EMN : \triangle EMN=AE-EL$, or $AMND : \triangle EMN=AL : EL$...(I).

$\triangle EMN : \triangle EBC=ME^2 : :BE^2=KE^2 : GE^2=AE\times EL : AE\times EK=EL : EK$.

$\therefore \triangle EMN-\triangle EBC : \triangle EMN=EL-EK : EL$, or $MBCN : \triangle EMN=LK : EL$...(II).

Comparing (I) and (II), $AMND : MBCN=AL : LK=1 : 1$.

*The reading of this problem has been slightly changed to correspond to the figure. ED. F.